REPORT No. 287

THEORIES OF FLOW SIMILITUDE

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SUMMARY

The laws of comparison of dynamically similar fluid motions are derived by three different methods based on the same principle and yielding the same or equivalent formulas. In this report prepared for publication by the National Advisory Committee for Aeronautics, in June, 1927, are outlined the three current methods of comparing dynamically similar motions, more especially of fluids, initiated respectively by Newton, Stokes (or Helmholtz), and Rayleigh. These three methods, viz., the integral, the differential, and the dimensional, are enough alike to be studied profitably together. They will presently be treated in succession then compared.

INTRODUCTION

Geometrically similar figures.—If two figures are geometrically similar, they have a constant scale ratio

$$l/l_1 = a \qquad (1)$$

where l, l_1 are any two homologous lengths. If x, x_1 etc., are homologous point coordinates for the figures, $x/x_1 = y/y_1 = z/z_1 = a$.

Geometrically similar motions.—Two similar configurations perform geometrically similar motions when their homologous points trace similar paths in proportional times; that is, in times t, t_1 having any arbitrary ratio b, the same for all homologous path segments. Thus v, v_1 being corresponding path speeds,

$$t/t_1 = b \qquad v/v_1 = lt_1/l_1t = a/b \qquad \dot{v}/\dot{v}_1 = lt_1^2/l_1t^2 = l_1v^2/lv_1^2 = a/b^2 \qquad (2)$$

where l/l_1 is the scale ratio of homologous moving parts, path segments, radii of curvature, etc. Since by (2) the ratio l_1v^2/lv_1^2 of accelerations normal to the path elements equals $\dot{v}/\dot{v}_1$ along them, the resultant accelerations, j, j_1 bear the same ratio and are alike directed. The constant ratios l/l_1, t/t_1, v/v_1, $\dot{v}/\dot{v}_1$ all may be different; only two can be independent, as (2) shows.[1]

Dynamically similar systems.—Let the homologous elements of two similar configurations in similar motion be masses m, m_1 having the constant ratio

$$m/m_1 = c = \rho l^3/\rho_1 l_1^3 \qquad (3)$$

ρ, ρ_1 being their densities; then, to keep their motions similar, all corresponding impressed forces must be in constant ratio and like direction.[2] For since these elements have resultant accelerations $j, j_1 \propto v^2/l, v_1^2/l_1$, their resultant impressed forces R, R_1 have the ratio

$$R/R_1 = mj/m_1j_1 = \rho l^2v^2/\rho_1 l_1^2 v_1^2 \qquad (4)$$

which is constant throughout, since ρ/ρ_1, l/l_1, v/v_1 are so. Further, the accelerations j, j_1 are alike directed; so then must be R, R_1. So, too, the corresponding forces on large homologous

[1] Were the paths similar irrespective of describing time, the motions still would be geometrically similar, but not as defined here and in usual writings on similitude. The geometrically similar motions here treated are kinematically similar because they trace similar paths in proportional times.

[2] That is, their magnitudes are in constant ratio and their lines of action similarly located in the two systems, though the systems themselves may be neither simultaneous nor alike oriented in space.

parts must be in constant ratio and like direction, as appears on compounding those on their constituent elements.[3] Also by the argument for (4) the constituents P, Q, P_1, Q_1, etc., of R, R_1, such as weight, pressure, friction, etc., must be in constant ratio and like direction, viz, P to P_1, Q to Q_1, etc. In fact for homologous elements they are concurrent and have similar force polygons. Hence

$$R/R_1 = \rho l^2 v^2/\rho_1 l_1^2 v_1^2 = P/P_1 = Q/Q_1 = \text{etc} \qquad (4_1)$$

Such systems are dynamically similar and have (1), (2), (4) as their conditions or criteria of similarity.

By (4) when ρ/ρ_1, l/l_1, v/v_1 are assumed constant R/R_1 is found constant. So, too, if ρ/ρ_1, l/l_1, R/R_1 are constant, v/v_1 is constant, and the motions are similar. Fixing either three of these ratios determines the fourth. Thus, premised initial similarity, similar mass systems in similar motion are similarly forced; conversely similar mobile mass systems similarly forced similarly move. In either case the systems are dynamically similar.

Summation of impressed forces.—The resultant forces R, R_1 at homologous elements have the components

$$\left.\begin{aligned} mj_x &= P_x + Q_x + \text{etc.} \\ m_1 j_{1x} &= P_{1x} + Q_{1x} + \text{etc.} \end{aligned}\right\} \qquad (5)$$

with like values for the y, z directions. These equations may be compared with (17), where the magnitudes of P, Q, etc., not merely their ratios, have definite expression; also with (13), where the magnitudes have only proportionate expression.

DYNAMICALLY SIMILAR FLOWS

A) NEWTONIAN OR INTEGRAL METHOD

Definition.—Fluid streams that everywhere satisfy (1), (2), (4) are dynamically similar systems, with similar flow fields and boundaries; hence are comparable in their corresponding characteristics.

Classification of chief force ratios.—As before, the ratio of the acceleration forces on homologous parts of such systems must be the same throughout and must equal severally the ratios of the corresponding impressed forces. The following table exhibits the chief ratios of present interest. Their proof follows the table. For all homologous elements the ratios g/g_1, ρ/ρ_1, μ/μ_1 are assumed constant, μ denoting viscosity.

TABLE I

RATIO OF CORRESPONDING FORCES ON HOMOLOGOUS FLUID ELEMENTS

Ratio of acceleration forces mj/m_1j_1	Ratio of impressed forces		
	Gravitational mg/m_1g_1	Pressural, $l^3\frac{\partial p}{\partial l}\Big/l_1^3\frac{\partial p_1}{\partial l_1}$	Viscous $\mu\partial v/\partial l=\mu_1\partial v_1/\partial l_1$
$\rho l^2v^2/\rho_1l_1^2v_1^2$	$g\rho l^3/g_1\rho_1 l^3_1$	$\rho l^2v^2/\rho_1l_1^2v_1^2$, for incompressible fluid; $\kappa l^2/\kappa_1 l_1^2$, for elastic fluid	$\mu lv/\mu_1l_1v_1$.

Proof of force ratios.—The ratio in column 1 has been proved; that in column 2 is obviously true.

To prove column 3, the pressure force on any small volume of frictionless fluid, being proportional to volume times along-stream pressure gradient, varies as $l^3.\partial p/\partial l$, as is well known, where $\partial p/\partial l \propto \partial(\rho v^2)/\partial l$. Hence for ρ constant the resultant pressure force varies as ρl^2v^2; and for ρ variable $\partial p/\partial l = \kappa/\rho.\partial\rho/\partial l$, by hydrostatics; that is, the pressure force varies as κl^2, where κ is the bulk modulus. One recalls that $\kappa/\kappa_1 = \rho c^2/\rho_1 c_1^2$ where c, c_1 are the speeds of sound in the fluids under the actual working conditions.

[3] Newton, reference 1, proves this theorem verbally without using symbols. A different symbolic treatment is given by Sir Richard Glazebrook in reference 2.

If $\partial v/\partial l$ is the rate of distortion in any fluid element, the entailed force on it varies as $l^2.\mu\partial v/\partial l \propto \mu l v$; hence the ratio in column 4.

Examples of similar flow conditions.—Granted kinematic similarity, when the impressed forces are as in Table I the general conditions (4_1) for dynamic similarity are

$$\frac{\rho l^3 v^2}{\rho_1 l_1^2 v_1^2} = \frac{g\rho l^3}{g_1\rho_1 l_1^3} = \frac{\mu l v}{\mu_1 l_1 v_1} = \begin{cases} \rho l^2 v^2/\rho_1 l_1^2 v_1^2, \text{ for incompressible fluids} \\ \kappa l^2/\kappa_1 l_1^2, \text{ for elastic fluids} \end{cases} \qquad (6)$$

where only the ratios of predominant forces are to be retained. A few examples will illustrate.

(α) Thus, if weight is the only dominant impressed force, the motions are dynamically similar when the first ratio in (6) equals the second, viz, when

$$gl/v^2 = g_1 l_1/v_1^2 \qquad (7)$$

which is the well-known Reech and Froude "law of corresponding speeds."

(β) If weight and elasticity are negligible, the first ratio in (6) is equated to the third, giving

$$\frac{\nu}{lv} = \frac{\nu_1}{l_1 v_1} \qquad (8)$$

which is the familiar Reynolds's condition for similarity of motion of fluids. It applies to the motion of airships, submarines, skin friction planes, fluids in pipes, etc.

(γ) If there is considerable compression, while gravity and friction are negligible, the first term is equated to the lower fourth, giving

$$\frac{c}{v} = \frac{c_1}{v_1} \qquad (9)$$

which is Booth and Bairstow's condition for similarity.

(δ) If g, μ, c all are important, conditions (7), (8), (9) must coexist; if all are negligible, (6) gives $\rho l^2 v^2/\rho_1 l_1^2 v_1^2 = \rho l^2 v^2/\rho_1 l_1^2 v_1^2$, that is, all flows with similar boundaries are similar, whatever the densities and velocities.[4]

Reactions in similar flows.—If P, P_1 are corresponding reactions of a craft and its model under conditions (7), $P/P_1 = g\rho l^3/g_1\rho_1 l_1^3$, whence

$$P = N_1\, g\rho l^3 \qquad (10)$$

where $N_1 = P_1/g_1\rho_1 l_1^3$ is a dimensionless coefficient, say, given by model tests.

If g, κ are negligible, conditions (8) obtain, and $P/P_1 = \mu l v/\mu_1 l_1 v_1$, or $P/P_1 = \rho l^2 v^2/\rho_1 l_1^2 v_1^2$, whence

$$P = N_2 \mu l v, \text{ or } P = C\rho l^2 v^2 \qquad (11)$$

where $N_2 = P_1/\mu_1 l_1 v_1$, $C = P_1/\rho_1 l_1^2 v_1^2$, both dimensionless coefficients.

If g, μ are negilgible, and compression important, $P/P_1 = \kappa l^2/\kappa_1 l_1^2$; hence

$$P = N_3 \kappa l \qquad (12)$$

where $N_3 = P_1/\kappa_1 l_1^2$, and conditions (9) prevail.

Let the Ps be all lifts or all drags or other like directed forces. Then, if g, μ, κ all are important together, the total of such reactions on the craft is

$$R = N_1 g\rho l^3 + N_2 \mu l v + N_3 \rho c^2 l^2 = \rho l^2 v^2 f(gl/v^2, \nu/lv, c/v) \equiv N\rho l^2 v^2 \quad {}^{5} \qquad (13)$$

got by summing (10), (11), (12), using $\kappa = \rho c^2$, then factoring off $\rho l^2 v^2$. One notes that (13) can be written: Total reaction = gravitational + frictional + pressural.

The validity of (13) was premised on dynamic similarity of motion of the craft and its model, as defined by the simultaneous conditions (7), (8), (9). That is,

$$N_1 gl/v^2 + N_2 \nu/lv + N_3 c^2/v^2 = f(gl/v^2, \nu/lv, c/v) = f(g_1 l_1/v_1^2, \nu_1/l_1 v_1, c_1/v_1) = N \qquad (14)$$

[4] (7) has the alternative form $g/j = g_1/j_1$; (8) the alternative $f/p = f_1/p_1$, where , f_1, p, p_1 are corresponding frictions and pressures per unit area of homologous surface elements δS, δS_1. With f, $f_1 \perp$ to p, p_1 the resultant stresses have slopes f/p, f_1/p_1 to the normals at δS, δS_1. The "laws" (7), (8) (9) are but corollaries of (6) or (4_1).

[5] More conventionally one writes $R = \rho l^2 v^2 f_1\,(v^2/gl, \nu/lv, v/c)$.

Alternatively (13) can be written

$$R=\mu lv\ f'(gl/v^2,\ \nu/lv,\ c/v)=N'\mu lv \qquad (13_1)$$

found by factoring off μlv and rearranging the result.

Writers sometimes say (13) shows that R varies as ρl^2v^2; they can as well say $R\propto \mu lv$ by (13_1). The first statement is true for conditions making f constant; the second for f' constant. (Fig. 1.)

Dynamic scale and scale effect.—If $N_1(=P_1/g_1\rho_1l_1^3)$, found from a model test; is plotted against g_1l_1/v_1^2 $(=gl/v^2)$ the graph is directly applicable to computing the full-scale reaction (10); similarly for the graph of N_2 against ν/lv and N_3 against c/v.

In such plots the dimensionless argument, say, ν/lv, is treated as a single independent variable. The graph is the same whether ν varies alone or l alone or v alone, or if two or three vary together. If N_2 varies as $(lv/\nu)^n$, it varies as l^n, v^n, ν^{-n}. The effect, for instance, of varying l can be learned by varying v or ν in the model test, and so for N_1, N_3.

One calls the independent dimensionless argument gl/v^2 the dynamic scale for the motion (α), and the variation of N_1 with scale the scale effect. Similarly ν/lv is the dynamic scale for

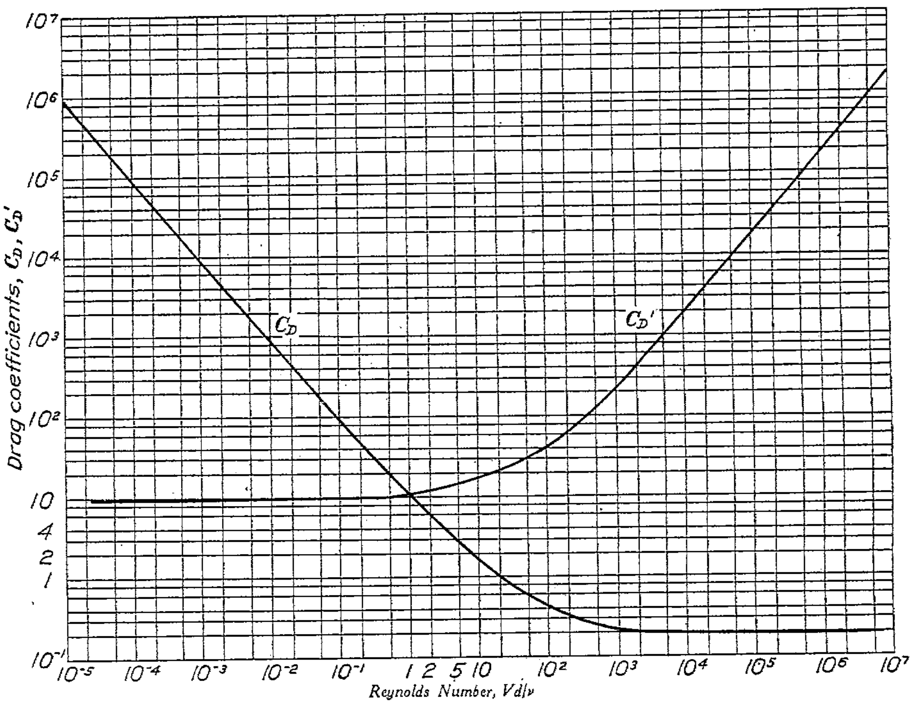

FIG. 1.—Drag coefficients $C_D=D/\rho V^2d^2$, $C_D'=D/\mu Vd$, plotted against Vd/ν for a sphere in uniform translation through a viscous fluid. Data given in N. A. C. A. Report No. 253. Either graph can be plotted from the other, since $C_D'/C_D=Vd/\nu$

motion (β), and l/c for (γ). No doubt the term "scale effect" originally meant the effect of changing the linear scale ratio l/l_1, then was extended to mean the effect of changing some more complex argument, such as gl/v^2, ν/lv, c/v, etc., now called the dynamic scale. The simpler scale ratios l/l_1, t/t_1, p/p_1, etc., are called scales of length, time, pressure, etc.

The more complex reaction (13) is a function of three dynamic scales, shown in parentheses. The scale effect here is the variation of N or R with one or more of the dynamic scales, or independent arguments gl/v^2, ν/lv, c/v. But for the particular case g, ν, $c=o$, as for a perfect liquid unaffected by gravity, N_1, N_2, N_3 are constant and have straight-line graphs when plotted against their scales. Then (13) gives $R=$const. times ρl^2v^2.

Generally, therefore, for dynamically similar fluid motions a dynamic scale is any one of the independent dimensionless arguments in the formula for the fluid reaction; the scale effect is the variation of such reaction due to variation of the arguments.

Arbitrary and derived scale ratios.—As seen in the introduction, geometric similarity requires one constant scale ratio, say, l/l_1, for length; kinematic similarity two scales, say, l/l_1, t/t_1, for length and time; dynamic similarity three, say, l/l_1, t/t_1, ρ/ρ_1. From these many others may be derived, say, v/v_1, $\dot{v}/\dot{v}_1$, p/p_1, ν/ν_1, etc., in case of fluids. Any three can be taken as determinative, then combined to form derived ratios, as exemplified in (2). For dynamic similarity we arbitrarily chose, at the outset, l/l_1, t/t_1, m/m_1, because length, time, and mass usually appear as basic. For the same service with fluid systems Helmholtz (reference 3) takes ρ/ρ_1, ν/ν_1. v/v_1, while other writers choose still other scales as fundamental.

Thus for geometrically similar fluid motions Helmholtz, assuming

$$\rho_1/\rho=r \qquad \nu_1/\nu=q \qquad u_1/u=v_1/v=w_1/w=n \qquad (15)$$

as given constants, thence derives the further ratios

$$x_1/x=y_1/y=z_1/z=q/n \qquad t_1/t=q/n^2 \qquad p_1/p=n^2rp+\text{const} \qquad (16)$$

for use in comparing the differential equations of motion of the two fluids.

(B) DIFFERENTIAL METHOD

The conditions (6) for dynamically similar motion of two fluids can also be derived from the standard differential equations of motion of such fluids, viz, from

$$\left.\begin{aligned}\rho\dot{u}&=\rho g_x-c^2\frac{\partial\rho}{\partial x}+\mu\Delta^2u+\frac{1}{3}\mu\frac{\partial\theta}{\partial x}\\ \rho_1\dot{u}_1&=\rho_1g_{1x}-c_1^2\frac{\partial\rho_1}{\partial x_1}+\mu_1\Delta^2u_1+\frac{1}{3}\mu_1\frac{\partial\theta_1}{\partial x_1}\end{aligned}\right\} \qquad (17)$$

with like expressions for the y, z and y_1, z_1 directions.[6] For if the motions are dynamically similar corresponding terms, all being forces, must have the same ratio,

$$\frac{\rho\dot{u}}{\rho_1\dot{u}_1}=\frac{\rho g_x}{\rho_1g_{1x}}=\frac{c^2}{c_1^2}\frac{\partial\rho/\partial x}{\partial\rho_1/\partial x_1}=\frac{\mu\Delta^2u}{\mu_1\Delta^2u_1}=\frac{\mu\partial\theta/\partial x}{\mu_1\partial\theta_1/\partial x_1}$$

Expressing these ratios in finite dimensions gives

$$\frac{\rho x_1u^2}{\rho_1xu_1^2}=\frac{\rho g_x}{\rho_1g_{1x}}=\frac{\rho x_1c^2}{\rho_1xc_1^2}=\frac{\mu x_1^2u}{\mu_1x^2u_1}$$

which multiplied by x^3/x_1^3 become the relations (6) for the x direction, viz,

$$\frac{\rho x^2u^2}{\rho_1x_1^2u_1^2}=\frac{g\rho x^3}{g_1\rho_1x_1^3}=\frac{\rho x^2c^2}{\rho_1x_1^2c_1^2}=\frac{\mu xu}{\mu_1x_1u_1}$$

Thus the differential method yields the same result as the Newtonian. It is Newton's method in Stokes's shorthand, except that Stokes would first write the forces, then their ratio; Newton would write their ratio directly. But to write their ratio one must know approximately their nature and analytic expression.

Helmholtz reverses the above argument. Assuming the relations (15), (16), he says they transform the first of (17) into the second, omitting the g terms. Hence he infers that model data serve to predict the hydrodynamic behavior of full-scale craft when the relations (15), (16) are maintained.

Diverse and sundry treatments of this topic are found in references 8, 9, 10.

(C) DIMENSIONAL METHOD

Mechanical units.—In mechanics three measuring units, say, of length, mass, time$\equiv L$, M, T, arbitrarily taken as fundamental, are combined in various powers to form other kinds called derived units, such as $U=AL^xM^yT^z$, A being constant, and x, y, z $>$ 1. Table II illustrates. These two classes of units, viz, fundamental and derived, serve to measure mechanical quantities of every kind, such as length, speed, torque, etc. Thus any mechanical quantity R, if a function of n others, all differing in kind, can be written

$$\mathrm{R}=\Sigma MQ_1^aQ_2^b\ldots Q_n^m \qquad (18)$$

where M, a, $b\ldots m$ are pure numbers, and Q may involve either fundamental or derived units. Table II gives examples.

[6] Here $\Delta^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}$; $\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$; etc., for Δ^2u_1, θ_1.

Basic formula.—To be valid for all unit systems (18) must be dimensionally homogeneous.[7] Then it can be written, if $[R]=[Q_1^a Q_2^b \ldots Q_n^m]=[P]$,

$$R=\Sigma NP \qquad (19)$$

where N is a pure number. This is the basic formula of dimensional theory, and is most general when the Ps are all the separate independent Q products that can be formed having the dimensions of R.[8]

Homogeneous products.—To find these P products, we first multiply any Q triad, say $Q^x Q^y Q^z$, by each remaining Q successively, and equate to $[R]$ the dimensions of each resulting product. The ensuing n–3 equations, with tentative exponents a, b, c, are

$$[P_1]=[Q_1^a Q_2^b Q_3^c Q_4]=[R] \qquad [P_2]=[Q_1^d Q_2^e Q_3^f Q_5]=[R], \text{ etc} \qquad (20)$$

Now replacing $[R]$ and $[Q]$ by their values in L, M, T and solving (20) for a, b, c ----, gives P_1, P_2, ----. The following example illustrates. For a rigorous analysis special works on dimensional theory may be consulted (references 4, 5).

Reactions in similar flows.—If the reaction R of a body in a fluid stream depends solely on ρ, l, v, g, μ, c, or density, size, speed, weight, viscosity, elasticity, all the separate independent products having the dimensions of R possible to make with them amount to 6–3, say, P_1, P_2, P_3. To form these, we take any triad $\rho^x l^y v^z$ of the six independent quantities and multiply it successively by the remaining ones g, μ, c, giving

$$P_1=\rho^a l^b v^c . g \qquad P_2=\rho^d l^e v^f . \mu \qquad P_3=\rho^g l^h v^i . c$$

and equate the dimensions of each product to $[R]=[ML/T^2.]$ The first yields, by Table II,

$$(M/L^3)^a . L^b . (L/T)^c . L/T^2 = ML/T^2$$

each unit having the same aggregate exponent in both terms. On equating the indices of L, M, T successively this gives

$$-3a+b+c+1=1 \qquad a=1 \qquad -c-2=-2$$

Thus $a=1$, $b=3$, $c=0$, whence $P_1=\rho l^3 g$. A like procedure gives $d=0$, $e=1=f$, whence $P_2=lv\mu$. Similarly $g=1$, $h=2$, $i=1$, whence $P_3=\rho l^2 vc$.

By (19) the reaction now is

$$R=N_1\,\rho l^3 g+N_2 \mu l v+N_3 \rho l^2 vc=\rho l^2 v^2 f(gl/v^2,\ \nu/lv,\ c/v) \qquad (13)$$

which is a general resistance equation for the specified dynamical conditions, viz, that R is a function solely of ρ, l, v, g, μ, c. From this (13_1) also is found as before.

By (13), if the arguments in parentheses are given any specific values the same for model and full scale, f is the same for both; hence

$$R=N\rho l^2 v^2$$

where $N=f(g_1 l_1/v^2,\ \nu_1/l_1 v_1,\ c_1/v_1)$, the same as by Newton's method.

COMPARISON OF THE THREE METHODS

In the foregoing text the same criteria for dynamical similarity in two flow systems were found by three different methods of analysis—the Newtonian, the differential, and the dimensional. In each the physical quantities governing the flow were premised from experience. Thence were found the ratios of corresponding impressed forces of each kind on homologous parts of the fluid, viz, weight ratio, pressure ratio, etc. These ratios, by definition of dynamic similarity, must each equal the ratio of the resultant acceleration forces on those parts; viz, the ratio $mj/m_1 j_1$.

By Newton's method we directly equated the ratio of these acceleration forces to the several ratios of corresponding impressed forces, thus obtaining specific conditions for dynamical

[7] That is, all terms of (18) must comprise the same fundamental units, each having a constant aggregate exponent throughout the equation.

[8] Dividing (19) by R gives $\psi(\pi_1, \pi_2 \ldots \pi_i)=0$, the πs being dimensionless products. This is Buckingham's π theorem (reference 4).

similarity, and formulas for the reactions of any fluid system in terms of those of its model. By the second method we first wrote the differential equations for the two fluid motions assumed dynamically similar, then equated the ratios of corresponding terms, thus obtaining the same result as before. By the third method we first equated the unknown reaction R on one fluid element to the sum of all the terms we could form from the flow-governing quantities arranged in power products each having the dimensions of R. Doing the same for the homologous element, then taking the ratio of the R forces, gave the same reaction formula as found by the other methods.

At first sight the dimensional process seems to be a routine algebraic operation requiring less knowledge than is needed for the two other methods. In reality all three demand adequate judgment of the kind of physical quantities governing the motion, and their comparative importance. In all three cases the assumed physical agencies are the same, the terms in the dynamic equations are analogous, and the final working formulas are the same or equivalent. In all, the derived working formula contains a dimensionless coefficient that is not deduced theoretically, but is to be found from model tests, then applied to full-scale apparatus operating under dynamically similar conditions. In all, the "laws of comparison" are merely expressions of equality of like dynamic scales, viz, equality of the ratio of the acceleration forces to the corresponding ratios of the dominant impressed forces.

TABLE II

QUANTITIES EXPRESSED IN BASIC UNITS OF LENGTH, TIME, AND MASS, L, T, M

The "dimensions" of a physical quantity are the degrees of the fundamental units in its formula. Thus the dimensions of an acceleration, which are symbolized by $[L\ T^{-2}]$, are 1 in length and −2 in time. Commonly the brackets are omitted from such simple L, T, M expressions not containing other symbols. The dimensions of a force are MLT^{-2}, viz, 1 in mass, 1 in length, −2 in time; the dimension of an angle, a sine, cosine, tangent, etc., is $L.L^{-1}$, that is zero. Logarithms in physical equations operate only on dimensionless quantities, such as pure numbers or ratios of like physical quantities; hence are dimensionless.

A derived unit, being formed of powers of fundamental units, has the form $U=AL^xM^yT^z$, with dimensions $L^xM^yT^z$. Thus a force $F=ms/t^2=AMLT^{-2}$, where m, s, t are mass, length, time in any convenient units. Its dimensions are written $[F]=[ms/t^2]=[AMLT^{-2}]=MLT^{-2}$.
In homogeneous equations all terms have the same dimensions, that is, the same aggregate exponent for each basic unit. Thus in the last equation of Table II each term has the dimensions $ML^{-1}T^{-2}$: for $[\rho v^2]=ML^{-3}.L^2T^{-2}=ML^{-1}T^{-2}$, and $[\mu v/a]=ML^{-1}\ T^{-1}.LT^{-1}/L=ML^{-1}\ T^{-2}$. In the familiar projectile formula $v=gt+c$, $[gt]=LT^{-2}.T=LT^{-1}=[v]=[c]$, where $c\ (=v_0)$ is a velocity.

Kind of quantity	Symbol. Formula	Dimensions of each term
	Derived units, $U=AL^xM^yT^z$	
Area, surface	$S=l^2$	L^2.
Volume	$\tau=l^3$	L^3.
Angle	$e=s/r=$ arc $\div$ radius	L^0.
Linear velocity	$u=x/t=dx/dt=\dot{x}$	LT^{-1}.
Linear acceleration	$j=u/t=du/dt=d^2x/dt^2$	LT^{-2}.
Angular velocity	$\omega=\theta/t=d\theta/dt=du/dy$	T^{-1}.
Angular acceleration	$\alpha=\omega/t=d\omega/dt=\theta/t^2$	T^{-2}.
Density	$\rho=m/\tau=m/l^3$	ML^{-3}.
Force, thrust	$F=mj=ms/t^2$	MLT^{-2}.
Torque, moment	$Q=Fl$	ML^2T^{-2}.
Pressure, friction	$p=F/S,\ f=F/S$	$ML^{-1}T^{-2}$.
Work, energy, potential	$W=Fs$	ML^2T^{-2}.
Power, activity	$P=Fu$	ML^2T^{-3}.
Viscosity	$\mu=f\div du/dy=f/\omega$	$ML^{-1}T^{-1}$.
Kinematic viscosity	$\nu=\mu/\rho$	L^2T^{-1}.
Flux of fluid	$\psi=\int q_n dS$	L^3T^{-1}.
Velocity potential	$\varphi=-\int q_t ds$	L^2T^{-1}.
	Geometrical and mechanical equations, $R=\Sigma NP$	
Length of catenary	$s=\frac{c}{2}(e^{x/c}-e^{-x/c})=c\sin h\left(\frac{x}{c}\right)$	L, $=L.L^0$.
Area of ellipse	$S=\pi ab$	L^2,
Volume of frustum of cone	$\tau=\frac{1}{3}\pi h(r^2+rr'+r'^2)$	L^3, $=[L(L^2+LL+L^2)]$.
Period of simple pendulum	$t=2\pi\sqrt{l/g}$	T, $=\sqrt{L/LT^{-2}}$.
Mutual attrac. of two particles	$F=\kappa mm^1/r^2$, where $[\kappa]=L^3/MT^2$	LMT^{-2}, $=L^3M^{-1}T^{-2}.M^2L^{-2}$.
Strength of line source	$m=2\pi aq_n$	L^2T^{-1}, $=L.LT^{-1}$.
φ for source-sink in plane stream	$\varphi=ux+\sigma\log\frac{r_2}{r_1}$, where $[\sigma]=L^2/T$	L^2T^{-1}, $=[LT^{-1}.L+L^2T^{-1}\log L^0]$.
Acceleration of viscous particle	$\dot{u}=-\partial p/\rho\ \partial x+\nu\Delta^2u$	LT^{-2} $=[p/\rho.x+\nu u/x^2]$.
Nose pressure on falling droplet	$p_n=\rho v^2/2+1.5\mu v/a$	$ML^{-1}T^{-2}=[\rho v^2+\mu v/a]$.

SYMBOLS USED IN TEXT

l, l_1	Homologous lengths in similar figures.
x, x_1, etc.	Homologous coordinates.
t, t_1	Times of tracing homologous paths.
v, v_1	Corresponding path velocities.
$\dot{v}$, $\dot{v}_1$	Corresponding path accelerations.
j, j_1	Corresponding total accelerations.
m, m_1	Homologous masses.
a, b, c	Arbitrary numerical ratios l/l_1, t/t_1, m/m_1; also tentative exponents.
ρ, μ, ν, κ	Density, viscosity, kinematic viscosity, bulk modulus; ditto for ρ_1, μ_1, ν_1, κ_1.
$c=\sqrt{\kappa/\rho}$	Speed of sound in elastic fluid; ditto for c_1.
R, R_1	Resultant forces on mass elements m, m_1.
P, Q, etc.	Components of R; P_1, Q_1, etc., components of R_1.
p, f	Pressure and friction, per unit area; ditto for p_1, f_1.

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